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On the NCCS model of the quantum Hall fluid

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Abstract. Area non-preserving transformations in the non-commutative plane are introduced with the aim to map the $\nu = 1$ integer quantum Hall effect (IQHE) state on the $\nu = \frac{1}{2p+1}$ fractional quantum Hall effect (FQHE) states. Using the hydrodynamical description of the quantum Hall fluid, it is shown that these transformations are generated by vector fields satisfying the Gauss law in the interacting non-commutative Chern–Simons gauge theory, and the corresponding field-theory Lagrangian is reconstructed. It is demonstrated that the geometric transformations induce quantum-mechanical non-unitary similarity transformations, establishing the interplay between integral and fractional QHEs.

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The apparent similarity between integral and fractional quantum Hall effects challenges the search for the theoretical schemes that bridge the gap between them.

One of the physically appealing models is the Jain composite fermion (CF) picture [1], which links the FQHE of interacting electrons to the IQHE of composite fermions. The CF picture gains mathematical content in the Chern–Simons (CS) gauge theory: the principal role of the CS gauge field is to attach an even number of elementary magnetic flux quanta to each electron, converting it to a composite particle.

In the present note we argue that the CS vector potential generates a geometric map between areas occupied by quantum fluids of different densities. The quantum-mechanical outcomes of these geometric transformations are the operator similarity transformations relating the quantum characteristics of the integral and fractional QHE states.

Below we consider QHE states, constituting the so called Laughlin series, characterized by the filling factors:

$$\nu = \frac{n_e}{n_B} = \frac{1}{2p+1}, \quad p = 0, 1, 2, \ldots \tag{1}$$

Here

$$n_e = \lim_{N,\Omega\to\infty} \frac{N}{\Omega} \tag{2}$$

is the average electron density (N is the number of electrons, and Ω is the occupied 2-dimensional area), and $n_B = \frac{B}{2\pi}$ is the density of quantum states per Landau level.

The most important common feature of the $p = 0$ and $p \geq 1$ states is that all of them are incompressible quantum fluids comprising the lowest Landau level (LLL) electrons. In the case of IQHE the ground state is given by the Slater determinant ($a = 1, 2, \ldots, N$)

$$\Psi_0 \sim \prod_{a<b} (z_a - z_b) \mathrm{e}^{-\frac{B}{4}\sum_a |z_a|^2} \tag{3}$$

(complex coordinates and the symmetric gauge are used).

At the fractional values of $\nu < 1$, which correspond to a partially filled LLL, the ground state is given by the Laughlin wave function [2]

$$\Psi_p \sim \prod_{a<b} (z_a - z_b)^{2p} \Psi_0 \,. \tag{4}$$

Note an important detail: the wave functions (3) and (4) both correspond to N-particle systems exposed to one and the same magnetic field B, but occupying different areas:

$$\Omega = \frac{2\pi}{B} N$$

and

$$\Omega' = (2p+1)\Omega \,, \tag{5}$$

respectively.

The many-particle wave functions (3) and (4) satisfy the LLL condition

$$\hat{\pi}_{\bar{z}}(a)\Psi_{\mathrm{LLL}} \equiv -\mathrm{i}\left[\bar{\partial}_a + \frac{B}{4} z_a\right]\Psi_{\mathrm{LLL}}(\mathbf{x}_1, \ldots, \mathbf{x}_N) = 0 \,. \tag{6}$$

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Equation (6) may be interpreted as the quantum counterpart of the classical second class Dirac constraints [3]:

$$\pi_i(a) \equiv p_i(a) + A_i(\mathbf{x}_a) \approx 0\,, \tag{7}$$

where p_i are the canonical momenta, and $A_i(\mathbf{x}) = \frac{B}{2}\epsilon_{ik}x_k$ is the electromagnetic vector potential.

The constraints (7) may be found from the zero-mass Lagrangian [4]

$$L = -\sum_{a=1}^{N}\left[\dot{x}_i(a,t)\frac{B}{2}\epsilon_{ik}x_k(a,t)\right]. \tag{8}$$

We do not include the confining potential, assuming that it takes some constant value in Ω and affects only edge states. The corresponding classical dynamical equations are given by

$$\dot{x}_i(a,t) = 0\,. \tag{9}$$

In other words, classically the electrons are frozen, i.e. they occupy fixed positions:

$$x_{0i}(a,t) = a_i\,. \tag{10}$$

Assuming that the system behaves like a perfect fluid one can pass to the hydrodynamical description [5], i.e. one may consider the electron system as a continuous distribution of particles, occupying the area

$$\Omega = \int_D \mathrm{d}^2x\,. \tag{11}$$

Particles are labeled by a continuous variable – the comoving coordinate $\boldsymbol{\xi}$, which is introduced via the replacement $\mathbf{x}(a,t) \to \mathbf{x}(\boldsymbol{\xi},t)$. The Lagrange variables $\boldsymbol{\xi}$ are fixed by the conditions

$$\xi_i = x_i(\boldsymbol{\xi},0)\,. \tag{12}$$

The zero-mass hydrodynamical Lagrangian is given by

$$\mathcal{L} = \int_D \mathrm{d}^2\xi\rho_0\left[-\frac{B}{2}\epsilon_{ik}\dot{x}_i(\boldsymbol{\xi},t)x_k(\boldsymbol{\xi},t)\right], \tag{13}$$

and we suppose that (13) corresponds to the IQHE state with filling factor $\nu = 1$ and with a constant density, $\rho_0 = n_B$.

In accord with (10), for the real trajectories we have

$$\mathbf{x}_0(\boldsymbol{\xi},t) = \boldsymbol{\xi}\,, \tag{14}$$

and one may set $\boldsymbol{\xi} \in D$. Consequently the occupied area is related to the particle density by the equation

$$\Omega = \int_D \mathrm{d}^2\xi = \rho_0^{-1}\int_D \mathrm{d}^2\xi\langle\rho(\boldsymbol{\xi})\rangle\,, \tag{15}$$

where $\langle\rho(\boldsymbol{\xi})\rangle$ is the microscopic density corresponding to $\nu = 1$.

Now consider the second droplet occupying the area Ω', assuming that both systems contain the same quantity of fluid. The primed system is described by the Lagrangian

$$\mathcal{L}' = \int_{D'} \mathrm{d}^2\xi'\rho_0'(\boldsymbol{\xi}')\left[-\frac{B}{2}\epsilon_{ik}\dot{x}_i'(\boldsymbol{\xi}',t)x_k'(\boldsymbol{\xi}',t)\right] \tag{16}$$

under the conditions that

$$\Omega' = \int_{D'} \mathrm{d}^2\xi' = (2p+1)\int_D \mathrm{d}^2\xi\,,$$
$$N = \int_D \mathrm{d}^2\xi\rho_0 = N' = \int_{D'} \mathrm{d}^2\xi'\rho_0'(\boldsymbol{\xi}')\,. \tag{17}$$

In other words, the Lagrangian (16) corresponds to $\nu = \frac{1}{2p+1}$.

The transition to the primed system is realized by means of the map $D \to D'$ [6,7]:

$$\xi_i \to \xi_i' = F_i(\xi) = \xi_i + \theta\epsilon_{ik}f_k(\xi,0), \quad \theta = -\frac{1}{B}\,. \tag{18}$$

In (18) $f_k(\xi,t)$ is a time-dependent vector field (here and below vector indices are omitted when they are obvious).

The hydrodynamical variables of the primed system are defined as follows:

$$x_i'(\xi',t) = x_{0i}(\xi,t) + \theta\epsilon_{ik}f_k(\xi,t), \quad x_{0i}(\xi,t) \equiv \xi_i$$
$$\dot{x}_i'(\xi',t) = \theta\epsilon_{ik}\dot{f}_k(\xi,t)\,, \tag{19}$$

and the corresponding Lagrangian and the occupied area are given by

$$\mathcal{L}' = \int_D \mathrm{d}^2\xi\rho_0\left[\frac{1}{2B}\epsilon_{ik}f_i(\xi,t)\dot{f}_k(\xi,t)\right] \tag{20}$$

and

$$\Omega' = \int_D \mathrm{d}^2\xi\left[1 + \epsilon_{ik}F_{ik}(\xi,t)\right], \tag{21}$$

respectively.

In the latter expression

$$F_{ik}(\xi,t) = D_if_k(\xi,t) \equiv \partial_if_k + \frac{1}{2}\{f_i,f_k\}_{\mathrm{D}}$$
$$\equiv \partial_if_k + \frac{1}{2}\theta\epsilon_{mn}\frac{\partial f_i}{\partial\xi_m}\frac{\partial f_k}{\partial\xi_n}\,, \tag{22}$$

and taking into account the definition of the filling factor one gets

$$\int_D \mathrm{d}^2\xi\epsilon_{ik}F_{ik}(\xi,t) = 2p\Omega\,. \tag{23}$$

Now recall (15) and assume that one may convert the integral constraint (23) to the local equation

$$4\pi p\langle\rho(\xi,t)\rangle + \epsilon_{ik}D_if_k(\xi,t) = 0\,. \tag{24}$$

Introducing the Lagrange multiplier $f_0(\xi,t)$, the constraint (24) may be combined with (20), resulting in the Lagrangian

$$\mathcal{L}_{\rm S}=\int_D {\rm d}^2\xi\rho_0\left\{\frac{1}{2B}\epsilon_{ik}f_i(\xi,t)\dot{f}_k(\xi,t)\right.\\ \left.+\theta f_0(\xi,t)\Big[4\pi p\langle\rho(\xi,t)\rangle+\epsilon_{ik}D_if_k(\xi,t)\Big]\right\}. \quad (25)$$

In the thermodynamic limit ($N\to\infty$, $\Omega\to\infty$) the Lagrangian (25) is equivalent to

$$\mathcal{L}_{\rm SCS}=\int_{R^2}{\rm d}^2\xi\left[\langle\rho(\xi,t)\rangle f_0(\xi,t)\right.\\ \left.+\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}f_\mu(\xi,t)D_\nu f_\lambda(\xi,t)\right], \quad (26)$$

where the covariant curl is

$$D_\mu f_\nu=\partial_\mu f_\nu+\frac{1}{2}\{f_\mu,f_\nu\}_{\rm D},\quad (\mu,\nu=0,1,2). \quad (27)$$

Following Susskind [5], the symplectic CS Lagrangian $\mathcal{L}_{\rm SCS}$ (with the zero source term) may be interpreted as a truncation of the non-commutative Chern–Simons (NCCS) Lagrangian.

In what follows we will directly get the NCCS Lagrangian considering geometric mappings in the non-commutative space. Non-commutativity enters on replacing the Poisson brackets of two canonical variables by the Dirac brackets [3] generated by the constraints (7). In particular, the Dirac bracket

$$\{x(a),y(b)\}_{\rm D}=\theta\delta_{ab}=-\frac{1}{B}\delta_{ab} \quad (28)$$

leads to non-commuting coordinate operators:

$$[\hat{x}_i(a),\hat{x}_k(b)]={\rm i}\epsilon_{ik}\theta\delta_{ab}. \quad (29)$$

Notice that, due to the non-commutativity of the coordinates, the hydrodynamical variables ξ_i have to be replaced by non-commutative quantities and the Lagrangian (13) by its NC analogue,

$$\mathcal{L}_{\rm NC}=\int_D{\rm d}^2\xi\rho_0\left[-\frac{B}{2}\epsilon_{ik}\dot{x}_i(\xi,t)\star x_k(\xi,t)\right], \quad (30)$$

where

$$f(\xi)\star g(\xi)={\rm e}^{\frac{\rm i}{2}\theta\epsilon_{ik}\partial_i\partial'_k}f(\xi)\cdot g(\xi')|_{\xi'=\xi} \quad (31)$$

is the Groenewold–Moyal star product.

Correspondingly, instead of the transformation (18) one has to consider the operator homomorphism [8]

$$\hat{\mathcal{W}}[\xi_i]=\hat{\xi}_i\to\hat{\mathcal{W}}[F_i]. \quad (32)$$

Here, by

$$\hat{\mathcal{W}}[F_i]=\frac{1}{(2\pi)^2}\int{\rm d}^2p\int{\rm d}^2x{\rm e}^{-{\rm i}p_i(\hat{\xi}_i-x_i)}F_i(x) \quad (33)$$

is denoted the symbol of Weyl ordering.

The resulting Lagrangian for the primed system will be

$$\mathcal{L}'_{\rm NC}=\int_D{\rm d}^2\xi\rho_0\left[\frac{1}{2B}\epsilon_{ik}f_i(\xi,t)\star\dot{f}_k(\xi,t)\right], \quad (34)$$

where the field $f_i(\xi,t)$ has to satisfy the NC analogue of (23), with

$$F_{ik}(\xi,t)=\theta\int_{R^2}{\rm d}^2xD(\xi-x)\left[\partial_if_k(x)-{\rm i}f_i(x)\star f_k(x)\right]\\ \equiv\theta\int_{R^2}{\rm d}^2xD(\xi-x)f_{ik}(x,t). \quad (35)$$

Here

$$D(\xi-x)=\frac{1}{\pi\theta}{\rm e}^{-\frac{1}{\theta}(x_i-\xi_i)^2}, \quad (36)$$

and in the commutative limit (35) reduces to (22). The last expressions may be derived by taking into consideration the area transformation rule in the NC plane [8].

The NC analogue of the constraint equation (23) is given by

$$2p\Omega=\frac{4\pi p}{B}\int_D{\rm d}^2\xi\langle\rho(\xi)\rangle\\ =-\frac{1}{B}\int_D{\rm d}^2\xi\int_{R^2}{\rm d}^2xD(\xi-x)\epsilon_{ik}f_{ik}(x,t). \quad (37)$$

Recall that the density

$$\langle\rho(\xi,t)\rangle:=\int_{R^2}{\rm d}^2xD(\xi-x)\rho(x,t) \quad (38)$$

corresponds to $\nu=1$. The non-commutative version of the Gauss law (GL) looks like

$$4\pi p\rho(x,t)+\epsilon_{ik}f_{ik}(x,t)=0. \quad (39)$$

Introducing the Lagrange multiplier $f_0(\xi,t)$ for the constraint (39), we arrive at the Lagrangian

$$\mathcal{L}_{\rm NC}=4\pi p\theta\rho_0\int_D{\rm d}^2\xi\left[\rho(\xi)f_0(\xi)-\frac{\kappa}{2}\epsilon_{ik}\left(f_i\star\dot{f}_k-f_0f_{ik}\right)\right], \quad (40)$$

where $k^{-1}=4\pi p$. In the thermodynamical limit (40) is equivalent to the NCCS Lagrangian

$$\mathcal{L}_{\rm NCCS}=\int_{R^2}{\rm d}^2\xi\left[\rho(\xi,t)\star f_0(\xi,t)\right.\\ \left.+\frac{\kappa}{2}\varepsilon^{\mu\nu\lambda}f_\mu\star\left(\partial_\nu f_\lambda-{\rm i}\frac{2}{3}f_\nu\star f_\lambda\right)\right]. \quad (41)$$

Some comments are in order here. As we have already remarked, in the context of QHEs this kind of Lagrangian was first introduced in [5] on the basis of area preserving diffeomorphisms (APDs) in the commutative plane. In our case the underlying transformations realize mappings between different areas in the non-commutative plane, i.e. they belong to the class of area non-preserving transformations. Non-commutative APDs are represented by the NC gauge transformations

$$f_i(\xi,t) \to f_i(\xi,t) + \frac{\mathrm{i}}{2\theta}\epsilon_{ik}(\xi_k \star \lambda - \lambda \star \xi_k)\,, \qquad (42)$$

under which the GL (39) is invariant.

Hence, the area transformation rule in the non-commutative plane leads to the GL in the NCCS gauge theory. In complex notation $(z = \xi_1 + \mathrm{i}\xi_2)$ it looks like

$$\left(\bar\partial f_z - \partial f_{\bar z}\right) + \left(f_z \star f_{\bar z} - f_{\bar z} \star f_z\right) = 2\mathrm{i}\pi p \rho(\xi)\,. \qquad (43)$$

This non-linear equation simplifies in the holomorphic gauge $f_{\bar z} = 0$, reducing the GL to the equation

$$\bar\partial f_z(\xi) = 2\mathrm{i}\pi p\rho(\xi)\,. \qquad (44)$$

The solution to (44) is given by

$$f_z(\xi) = 2\mathrm{i}p\partial \int \mathrm{d}^2\xi' \ln(z-z')\rho(\xi') = \mathrm{i}S^{-1}\partial S\,, \qquad (45)$$

where the holomorphic function

$$S = \mathrm{e}^{2p\int \mathrm{d}^2\xi' \ln(z-z')\rho(\xi')}, \quad \bar\partial S = 0\,. \qquad (46)$$

One easily verifies that the geometric transformation (18) induces the operator homomorphism

$$\begin{aligned} W(\xi_i) \to W(\xi_i') &= W\left(S^{-1}\right) W(\xi_i) W(S) \\ &= W\left(\xi_i + \theta\epsilon_{ik} f_k(\xi)\right)\,. \end{aligned} \qquad (47)$$

The expression (47) is the non-commutative space analogue of the usual coordinate transformation. The latter may be associated with a transformation of the mean values:

$$\begin{aligned} \langle\Phi|W(\xi_i)|\Phi\rangle &\to \langle\Phi|W(\xi_i')|\Phi\rangle \\ &= \left\langle\Phi|W\left(S^{-1}\right)W(\xi_i)W(S)|\Phi\right\rangle\,. \end{aligned} \qquad (48)$$

As an alternative, one may attribute geometric transformations to the map in the Hilbert space $|\Phi\rangle \to |\Phi'\rangle = W(S)|\Phi\rangle$, $\langle\Phi| \to \langle\Phi|W(S^{-1})$, keeping the coordinate operators unchanged: $W(\xi_i) \to W(\xi_i)$.

In the NC plane the operators $\hat z$ and $\hat{\bar z}$ are realized by

$$W(z) \equiv \hat z = z, \quad W(\bar z) \equiv \hat{\bar z} = \frac{2}{B}\frac{\partial}{\partial z}$$

and

$$W(S) = S\,. \qquad (49)$$

Then the operator transformations

$$\begin{aligned} S^{-1}\hat{\bar z}S &= \hat{\bar z} + 2\mathrm{i}\theta f_z(z) \\ S^{-1}\hat z S &= \hat z \end{aligned} \qquad (50)$$

reproduce in operator form the map $\xi_i \to \xi_i'$.

Now one may go back to the quantum-mechanical picture. The reference, i.e. the $\nu = 1$ QHE state, is described by the wave function (3) and the dynamics is governed by the constraint equations (6). Assuming that for the microscopic density one has

$$\rho(\mathbf{x}) = \sum_a \delta(\mathbf{x} - \mathbf{x}_a)\,, \qquad (51)$$

one gets the result that the geometric transformation is realized by the vector field

$$f_z(\mathbf{x}_a) = 2\mathrm{i}p{\sum_b}' \frac{1}{z_a - z_b}, \quad f_{\bar z}(\mathbf{x}_a) = 0\,. \qquad (52)$$

The holomorphic function

$$S_p(z_1, \ldots, z_n) = \mathrm{e}^{2p\sum_{a<b}\ln(z_a - z_b)} = \prod_{a<b}(z_a - z_b)^{2p} \qquad (53)$$

generates the transformations

$$\begin{aligned} &z_a \to z_a, \\ &\frac{\partial}{\partial z_a} \to S_p^{-1}\frac{\partial}{\partial z_a}S_p = \frac{\partial}{\partial z_a} + 2p{\sum_b}'\frac{1}{z_a - z_b}\,. \end{aligned} \qquad (54)$$

In the alternative picture the transition from the $\nu = 1$ IQHE state to the $\nu = \frac{1}{2p+1}$ FQHE state is accomplished by the map

$$\Psi_p = \Psi_0 \to S_p\Psi_0 = \prod_{a<b}(z_a - z_b)^{2p+1}\mathrm{e}^{-\frac{B}{4}\sum_a |z_a|^2}\,, \qquad (55)$$

reproducing the Laughlin wave function (4). In parallel, the relevant quantum operators (like constraints or guiding center coordinates) have to undergo the similarity transformations

$$\hat{\mathcal{O}}_0 \to \hat{\mathcal{O}}_p = S_p \hat{\mathcal{O}}_0 S_p^{-1}\,. \qquad (56)$$

In particular, the LLL constraint remains invariant:

$$\hat\pi_{\bar z}(a) \to \hat\Pi_{\bar z}(a) = S_p \hat\pi_{\bar z}(a) S_p^{-1} = \hat\pi_{\bar z}(a)\,, \qquad (57)$$

and the LLL condition is preserved,

$$\hat\pi_{\bar z}\Psi_p = 0\,. \qquad (58)$$

Note that the similarity transformations (55) and (56) are induced by geometric mappings relating different quantum Hall droplets, and they establish a non-unitary equivalence between the integral $(p = 0)$ and fractional $(p > 1)$ quantum Hall states.

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